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The Newsvendor Problem with Fast Moving Items and a Compound Poisson Price Dependent Demand

Anna V. Kitaeva, Valentina I. Subbotina*, Oleg A. Zmeev

National Research Tomsk State University, Tomsk, Russia, (e-mail: kit1157@yandex.ru).

Abstract. We consider a single period single item inventory system when the demand is a compound Poisson process with price dependent intensity and continuous batch size distribution with known mean and variance, and the mean of the demand is large enough. Equations for retail price maximizing the expected profit under optimal order quantity are obtained and approximate solution is proposed. Approximate distribution of the large order's selling time is found and results of testing procedures are given.

Keywords: Compound Poisson, Pricing, Newsvendor, Fast Moving Items, Selling Time, Expected Profit.

1. INTRODUCTION AND PROBLEM STATEMENT

The newsvendor problem (NP) is one of the classical problems of inventory management; see, for example, Arrow, Harris, and Marshak (1951); Silver, Pyke, and Peterson (1998). It has been studied since the eighteenth century and widely used to analyse systems with perishable products in such different fields as, for example, health insurances, airlines, sports and fashion industries. And nowadays a lot of papers related to this problem are still being published, see reviews by Khouja (1999); Qin et al. (2011); a handbook editing by Tsan-Ming Choi (2014).

Originally the price is exogenous variable in the NP. Whitin (1955) was the first who analyzes the price-dependent demand and determines the order size and selling price simultaneously. The topic has been the theme of many papers; see reviews by Petruzzi and Dada (1999); Yano and Gilbert (2003); Chen and Simchi-Levi (2012).

Consider a supply chain consisting of a supplier, a buyer, and customers. The duration of the product's lifetime is T. At the beginning of a time period T the buyer purchases a quantity Q at a fixed price per unit d (wholesale price).

Let the demand be a Poisson process with price-dependent intensity $\lambda(c)$, where $c > d$ is a selling (retail) price per unit of the product, the values of orders (batch sizes) be i.i.d. continuous random variables with finite the first and second moments equals respectively a_1 and a_2. So we consider a stochastic static model.

We assume that the buyer operates with no capacity restrictions, and we do not consider the cost of leftovers utilization and lost sales. The aim is to maximize the expected profit and to find the distribution of the order's selling time.

Due to complexity of the problem we solve it for fast moving items. It gives us a possibility to use the normal approximation to distribution of demand and the diffusion approximation of demand process. In such framework we manage to obtain the equation for optimal price, see (8), and the distribution of the selling time, see (13) and (14), in closed form. The latter can be used to estimate the mean and standard deviation of demand in the case of unobserved lost sales.

Note that taking into account the approximation the price optimization's task can be solved in the framework of additive-multiplicative model of demand firstly presented by Young (1978).

Denote $X(t)$ a random customer demand at $[0,t]$, $p(\cdot)$ the probability density function of $X(T) = X$.

Expected profit for the buyer at the end of the period

$$S = -Qd + cQ\int_Q^\infty p(x)dx + c\int_0^Q xp(x)dx.$$

The buyer is interested in determining an optimal value of Q and then corresponding value of c by maximizing the expected profit. The characteristics of the duration of the order's selling are also of interest.

Obviously the first task has unique solution determined by the equation

$$\int_Q^\infty p(x)dx = \frac{d}{c}, \tag{1}$$

and corresponding profit

$$S_0 = c\int_0^Q xp(x)dx. \tag{2}$$

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The result is well known; see, for example, Silver, Pyke, and Peterson (1998).

So we need to know distribution of X. In general the distribution is very complicated and it is difficult to solve the NP in closed form.

Denote probability density of the batch sizes $f(\cdot)$. Consider, for example, the case of exponential distribution

$$f(x) = \frac{1}{a_1}\exp\left(-\frac{x}{a_1}\right), \quad x \ge 0. \tag{3}$$

It is easy to see that

$$p(x) = \delta(x)e^{-\lambda(c)T} + e^{-x/a_1-\lambda(c)T}\sqrt{\frac{\lambda(c)T}{a_1 x}}I_1\left(2\sqrt{\frac{\lambda(c)Tx}{a_1}}\right),$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind and first order, $\delta(\cdot)$ is the Dirac delta function.

Formula (1) can be written as

$$\sqrt{\lambda(c)T}\int_{Q/a_1}^{\infty}\frac{1}{\sqrt{u}}I_1\left(2\sqrt{\lambda(c)Tu}\right)e^{-u-\lambda(c)T}du = \frac{d}{c}. \tag{4}$$

Only numerical solution (4) with respect to Q is possible and the task of price optimization given Q becomes enough of an onus. So below we consider the approximate solution of the problem for fast moving items.

2. PRICE OPTIMIZATION FOR FAST MOVING ITEMS

Let us consider the case $\lambda(c)T >> 1$.

Taking into account $I_1(z) \sim \frac{1}{\sqrt{2\pi z}}e^z$ as $z \to \infty$; see Abramowitz and Stegun (1972), we receive from (4)

$$Q = a_1\lambda(c)T + \sqrt{a_2\lambda(c)T}\Psi\left(1-\frac{d}{c}\right), \tag{5}$$

where $\Psi(\cdot) = \Phi^{-1}(\cdot)$, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(\frac{-t^2}{2}\right)dt$.

Formula (5) is valid in general, i.e. for any distribution $f(\cdot)$ as $\lambda T \to \infty$, because according central limit theorem $\frac{X - a_1\lambda(c)T}{\sqrt{a_2\lambda(c)T}}$ converges in distribution to a standard normal random variable $N(0,1)$.

Formula (5) is also well-known as exact result for normally distributed demand. We know that the normal distribution provides a good empirical fit to observed demand data for fast moving items.

Now find the optimal retail price for order quantity Q defined by (5).

From (2) we get as $\lambda(c)T >> 1$

$$S_0 = c\left[a_1\lambda(c)T\Phi\left(\frac{Q-a_1\lambda(c)T}{\sqrt{a_2\lambda(c)T}}\right) - \right.$$
$$\left. -\sqrt{\frac{a_2\lambda(c)T}{2\pi}}\exp\left(-\frac{(Q-a_1\lambda(c)T)^2}{2a_2\lambda(c)T}\right)\right].$$

For optimal value Q defined by (5) we get

$$S_0 = c\left[a_1\lambda(c)T\left(1-\frac{d}{c}\right) - \sqrt{\frac{a_2\lambda(c)T}{2\pi}}\exp\left(-\frac{1}{2}\Psi^2\left(1-\frac{d}{c}\right)\right)\right].$$

It is obvious that $S_0 \ge 0$, and $S_0 \approx 0$ as $c \approx d$. Let the following natural condition holds: $c\lambda(c) \approx 0$ as $c >> 1$, then $S_0 \approx 0$ as $c >> 1$. It follows that maximum value of $S_0(c)$ exists.

Let $\lambda(c) = \lambda_0 F(c)$, where $F(\cdot)\downarrow$ is a twice differentiable function. Then

$$S_0 = a_1\lambda_0 T\left[F(c)(c-d) - \right.$$
$$\left. -c\sqrt{\frac{a_2 F(c)}{2\pi a_1^2\lambda_0 T}}\exp\left(-\frac{1}{2}\Psi^2\left(1-\frac{d}{c}\right)\right)\right]. \tag{6}$$

If $\lambda(c)T >> 1$ then considering only the first summand in (6) we get a “null” approximation c_0 to optimal value of retail price c^o

$$c_0 = \arg\max_c\left(F(c)(c-d)\right).$$

The approximation is defined by equation

$$c_0 + \frac{F(c_0)}{F'(c_0)} = d. \tag{7}$$

Price c_0 corresponds to the case when the number of orders and, consequently, the lot size are very large.

Taking derivative $\frac{dS_0}{dc}$ we receive the equation for c^o

$$F'(c)(c-d) + F(c) - \sqrt{\frac{a_2}{2\pi a_1^2\lambda_0 T}}\times$$
$$\times\left[\left(\sqrt{F(c)} + \frac{cF'(c)}{2\sqrt{F(c)}}\right)\exp\left(-\frac{1}{2}\Psi^2\left(1-\frac{d}{c}\right)\right) + \right.$$

$$+\sqrt{2\pi F(c)}\frac{d}{c}\Psi\left(1-\frac{d}{c}\right)\Bigg]=0. \qquad (8)$$

Let us consider $c^o = c_0 + \Delta c$, $\Delta c \sim 1/\sqrt{\lambda_0 T}$ as $\lambda_0 T \to \infty$, and let $c = c_0$ in the additional (second) part of (8) because the second part is proportional to $1/\sqrt{\lambda_0 T}$.

Using Taylor's expansion for the first part of (8)

$$F'(c)(c-d)+F(c)=\left[F''(c_0)(c_0-d)+2F'(c_0)\right]\Delta c+...$$

we get

$$\Delta c=\sqrt{\frac{a_2}{2\pi a_1^2\lambda_0 T}}\frac{1}{F''(c_0)(c_0-d)+2F'(c_0)}\times$$

$$\times\left[\left(\sqrt{F(c_0)}+\frac{c_0F'(c_0)}{2\sqrt{F(c_0)}}\right)\exp\left(-\frac{1}{2}\Psi^2\left(1-\frac{d}{c_0}\right)\right)+\right.$$

$$\left.+\sqrt{2\pi F(c_0)}\frac{d}{c_0}\Psi\left(1-\frac{d}{c_0}\right)\right]. \qquad (9)$$

For instance, consider a linear function

$$F(c)=1-a\frac{c-c_0}{d},\ a>0.$$

Then (7) can be written as follows $-\frac{a}{d}(c_0-d)+1=0$, and $c_0=d\left(1+\frac{1}{a}\right)$.

Note that $c_0 >> 1$ as $a \approx 0$. It is quite natural because here the intensity of the customer's orders does not depend on the retail price. And $c_0 \approx d$ as $a >> 1$. It is also quite expected.

From (9) we get

$$\Delta c=-d\sqrt{\frac{a_2}{2\pi a_1^2\lambda_0 T}}G(a),$$

where

$$G(a)=\frac{1}{2a}\left[\frac{1-a}{2}\exp\left(-\frac{1}{2}\Psi^2\left(\frac{1}{1+a}\right)\right)+\sqrt{2\pi}\frac{a}{1+a}\Psi\left(\frac{1}{1+a}\right)\right].$$

Plot of $G(a)$ is shown in Figure 1. If $a<1$, then $G(a)>0$, and it follows that $\Delta c<0$, so the retail price needs to be a little less than c_0. If $a>1$, then $G(a)<0$, and in this case $\Delta c>0$, so the retail price needs to be a little more than c_0.

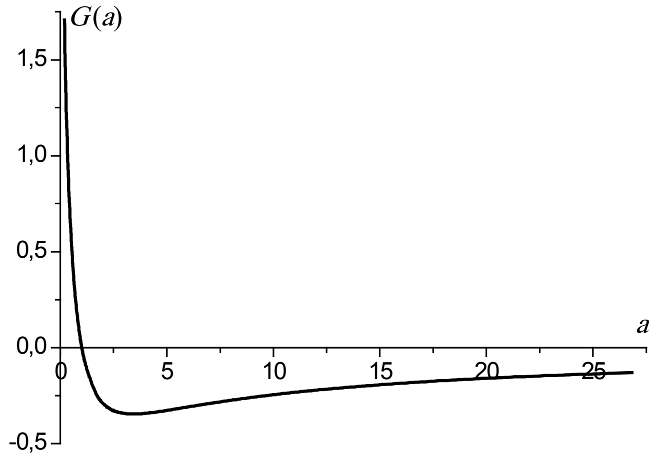

Fig. 1. Dependence Δc of a for linear function $F(\cdot)$.

3. DISTRIBUTION OF THE SELLING TIME

Let us find asymptotic distribution $u(\cdot)$ of the length of time τ it takes to sell the lot for the model under consideration. For exponential distribution of batch size the exact solution is possible

$$u(t)=\lambda e^{-\lambda t-Q/a_1}+\sum_{n=2}^{\infty}\frac{\lambda^n t^{n-1}}{(n-1)!}e^{-\lambda t}\frac{Q^{n-1}}{a_1^{n-1}(n-1)!}e^{-Q/a_1}=$$

$$=\lambda e^{-\lambda t-Q/a_1}\left[1+\sum_{s=1}^{\infty}\frac{1}{(s!)^2}\left(\frac{\lambda tQ}{a_1}\right)^s\right].$$

Taking into account following equation from Abramowitz and Stegun (1972)

$$1+\sum_{s=1}^{\infty}\frac{1}{(s!)^2}\left(\frac{\lambda tQ}{a_1}\right)^s=I_0\left(2\sqrt{\frac{\lambda tQ}{a_1}}\right),$$

we finally get

$$u(t)=\lambda e^{-\lambda t-Q/a_1}I_0\left(2\sqrt{\frac{\lambda tQ}{a_1}}\right),$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind and zeroth order.

Denote dimensionless quantity $Q/a_1=q$ and random value $\lambda\tau=\gamma$. Probability density of γ

$$p_\gamma(x)=e^{-x-q}I_0\left(\sqrt{2xq}\right).$$

Consider behavior of $p_\gamma(x)$ as $q>>1$. Let $\eta=\sqrt{\gamma}$. Probability density of η

$$p_\eta(x) = 2xe^{-x^2-q} I_0\left(2x\sqrt{q}\right).$$

Taking into account $I_0(z) \sim \frac{1}{\sqrt{2\pi z}} e^z$ as $z \to \infty$, we get

$$p_\eta(x) \sim \frac{1}{\sqrt{\pi}} e^{-(x-\sqrt{q})^2} \frac{\sqrt{x}}{\sqrt[4]{q}} \approx \frac{1}{\sqrt{\pi}} e^{-(x-\sqrt{q})^2}.$$

Here and below $g(x) \approx f(x)$ means that $\sup_x |f(x) - g(x)| \to 0$ as $q \to \infty$.

Let $\eta = \sqrt{q} + \xi$, where ξ is $N(0,1/2)$, then $\gamma = \eta^2 = q + 2\xi\sqrt{q} + \xi^2$. It follows $(\gamma - q)/\sqrt{2q}$ convergences to $N(0,1)$ as $q \to \infty$ and

$$\tau = \gamma/\lambda \sim N\left(\frac{Q}{a_1\lambda}, 2\frac{Q}{a_1\lambda^2}\right). \quad (10)$$

The result can be generalized to any distribution $f(\cdot)$, if we consider diffusion approximation of demand process $X(t)$

$$dX(t) = a_1\lambda dt + \sqrt{a_2\lambda} dw(t),$$

where $w(\cdot)$ is the Wiener process.

Diffusion methods have been applied to inventory models in a variety of domains to begin with the papers by Bather (1966) and Puterman (1975). In Kitaeva (2014) the diffusion approximation of stock level process has been used to find the steady-state distribution and to solve the problem of on/off control minimizing the variance of the process. In Kitaeva, Subbotina, and Zmeev (2014) theoretical justification of a diffusion approximation with some numerical results is given.

Denote $\tau(x)$ the first occurrence time of the crossing of level Q by $X(\cdot)$ given $X(0) = x$; $g(s,x) = E\left\{e^{-s\tau(x)}\right\} = \int_0^\infty e^{-sy} u(y|x) dy$ is the Laplace transform of conditional density $u(\cdot|x)$.

Consider

$$g(s,x) = E\left\{e^{-s(\Delta t + \tau(x+\Delta x))}\right\} = e^{-s\Delta t} E_{\Delta x}\{g(s, x+\Delta x)\},$$
$$\Delta x = X(t+\Delta t) - X(t),$$

where $E_{\Delta x}\{\cdot\}$ denote the expectation with respect to random value Δx.

Using Taylor's expansions we get

$$g(s,x) = (1 - s\Delta t) \times$$
$$\times E_{\Delta x}\left\{g(s,x) + \frac{\partial g(s,x)}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 g(s,x)}{\partial x^2}(\Delta x)^2\right\} + o(\Delta t) =$$
$$= g(s,x) + \left[-sg(s,x) + \frac{\partial g(s,x)}{\partial x} a_1\lambda + \frac{a_2\lambda}{2}\frac{\partial^2 g(s,x)}{\partial x^2}\right]\Delta t +$$
$$+ o(\Delta t).$$

It follows as $\Delta t \to 0$

$$\frac{\partial^2 g(s,x)}{\partial x^2} + 2\frac{a_1}{a_2}\frac{\partial g(s,x)}{\partial x} - \frac{2s}{a_2\lambda} g(s,x) = 0; \; g(s,Q) = 1. \quad (11)$$

Denote the value of interest $\tau(0) = \tau$, and $g(s,0) = g_\tau(s)$. From (11) we get

$$g_\tau(s) = \exp\left(\frac{a_1}{a_2}Q - \sqrt{\frac{a_1^2}{a_2^2} + \frac{2s}{a_2\lambda}}\,Q\right). \quad (12)$$

From (12) it follows $E\{\tau\} = \frac{Q}{a_1\lambda}$, and $Var\{\tau\} = \frac{a_2 Q}{a_1^3\lambda^2}$. The inverse Laplace transform of $g_\tau(s)$; see Bateman and Erdely (1954),

$$u(t) = \frac{Q}{\sqrt{2\pi a_2\lambda t^{3/2}}} \exp\left(-\frac{a_1^2\lambda}{2a_2 t}\left(t - \frac{Q}{a_1\lambda}\right)^2\right). \quad (13)$$

Consider the case $\frac{E\{\tau\}}{\sqrt{Var\{\tau\}}} = \sqrt{\frac{Qa_1}{a_2}} = \sqrt{q\frac{a_1^2}{a_2}} >> 1$, i.e. the lot size is large enough.

Then

$$u(t) \approx \sqrt{\frac{a_1^3\lambda^2}{2\pi a_2 Q}} \exp\left(-\frac{a_1^3\lambda^2}{2a_2 Q}\left(t - \frac{Q}{a_1\lambda}\right)^2\right). \quad (14)$$

Result (10) follows from (14) by taking into account that for exponential distribution $a_2 = 2a_1^2$.

The results of the Kolmogorov-Smirnov test for normality of selling time distribution reports an acceptance of the hypothesis of normality with a p-value more than 0.05 for $Q \geq 6$ under uniform distribution; $Q \geq 13$ under exponential distribution; $Q \geq 8$ under beta distribution. In Tables 1−3 some results are given.

The simulation setups are as follows:

1. Batch size has uniform distribution at [0, 6]. It follows that $a_1 = 3$ and $a_2 = 12$. The intensity λ equals 1.

2. Batch size has exponential distribution with $a_1 = 4$. It follows that $a_2 = 32$. The intensity λ equals 1.

3. Batch size has beta distribution with shape parameters $\alpha = 0.429$ and $\beta = 0.286$. It follows that $a_1 = 0.6$ and $a_2 = 0.5$. The intensity λ equals 4.

Table 1. Uniform distribution, $a_1 = 3$, $a_2 = 12$, $\lambda = 1$

Q	K-S statistic	p-value
5	0.136	0.048
7	0.124	0.091
30	0.089	0.403
90	0.068	0.741
92	0.056	0.916

Table 2. Exponential distribution, $a_1 = 4$, $a_2 = 32$, $\lambda = 1$

Q	K-S statistic	p-value
8	0.162	0.010
17	0.112	0.161
80	0.084	0.467
160	0.072	0.674
194	0.053	0.944

Table 3. Beta distribution, $a_1 = 0.6$, $a_2 = 0.5$, $\lambda = 4$

Q	K-S statistic	p-value
5	0.048	0.017
8	0.041	0. 062
80	0.033	0.214
110	0.023	0.613
130	0.019	0.835

4. CONCLUSION

First of all, considering approximation for fast moving items gives us a possibility to solve the price optimization problem for a compound Poisson demand with price dependent intensity in the NP framework.

Note, here we need to know only the first and second moments of a distribution of the demand. We can estimate parameters of interest $a_1\lambda T$ and $a_2\lambda T$ using two samples: the selling durations $t_1, t_2, \ldots,\ \forall i\ t_i \le T$, if there are lost sales, and the sizes of sales during time period T $x_1, x_2, \ldots,\ \forall i\ x_i \le Q$, if there are leftovers at the end of the period. We assume that the lot sizes are the same for each period and base the estimating procedure on the distributions obtained above, for details see Kitaeva, Subbotina, and Stepanova (2015).

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